Comprehensive Test on Analysis

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(1) Let μ be a **real signed** measure on $[0,1]$ with finite variation and $\mu([a,b]) = 0$ for any $0 \le a \le b \le 1$. Show that $\mu \equiv 0$.

(2) Let Ω be a bounded domain in $\mathbb{R}^d$.

   (a) If f is in $L^p(\Omega), p \ge 1$, then $f \in L^q(\Omega), 1 \le q \le p$.

   (b) Let $\{f_n\}$ be bounded in $L^p(\Omega), p > 1$. Furthermore, assume that $f_n \to f$ pointwise as $n \to \infty$. Prove that $f_n \to f$ (strongly) in $L^q(\Omega)$ for $\mathbf{1 \le q < p}$.

(3) Show that there is no non-negative (non-trivial) measure γ on an **infinite dimensional** Banach space with the property that $\gamma(aB) = a^\beta \gamma(B), a \ge 0$, and $\gamma(x + B) = \gamma(B)$ for all Borel set B and some constant $\beta > 0$. In short, prove that such a $\gamma \equiv 0$. [Hint: You can use Riesz's lemma: Let X be a Banach space with subspace Z, Y, and Y be a proper closed subspace of Z. Then for $\theta \in (0,1)$ there exits $z \in Z$ with $\|z\| = 1$ and $\inf_{y \in Y} \|z - y| \ge \theta$.]

(4) If f is a non constant entire function, prove that the image of f is dense in $\mathbf{C}$.

(5) Prove that $1/z$ is not the uniform limit of a sequence of polynomials on the annulus $\{z : 1 < |z| < 2\}$.

(6) Let $\lambda > 1$ and prove that the equation $\lambda - z - e^{-z}$ has exactly one zero on the right half plane $\{z : \Re z > 0\}$.

Following are the answers:

(1) By Hahn decomposition we can write $\mu = \mu^+ - \mu^-$. Then we have $\mu^+([a,b]) = \mu^-([a,b])$. But closed intervals generate the Borel σ-algebra on $[0,1]$. Thus $\mu^+ = \mu^-$ implying $\mu \equiv 0$.

(2) It is easy to see that for any constant M, $f_n \chi_{\{|f_n| < M\}} \to f \chi_{\{|f| < M\}}$ almost surely. In fact, this convergence is in any L^q using dominated convergence theorem and the property that Ω is bounded. So it is enough to show that $\sup_n \|f_n \chi_{\{|f_n| \ge M\}}\|_{L^q} \to 0$ as $M \to \infty$. By assertion we have $\sup_n \|f_n\|_{L^p} < K$. Then

$$\int_\Omega |f_n|^q \chi_{\{|f_n| \ge M\}} \le \frac{1}{M^{p-q}} \int_\Omega |f_n|^p \le \frac{K^p}{M^{p-q}}.$$

(3) It is enough to show that the unit ball B_1 has zero measure. If not, let $\gamma(B_1) > 0$. Now use Riesz's lemma to generate a sequence x_n so that $\|x_n\| = 1$ and $\|x_i - x_j\| > \frac{1}{2}$. Then the balls $x_i + \frac{1}{4}B_1$ are disjoint and included in B_2. By the scaling property $\gamma(x_i + \frac{1}{4}B_1) = \frac{1}{4^\beta}\gamma(B_1) > 0$. This shows $2^\beta\gamma(B_1) = \gamma(B_2) \geq \sum_n \gamma(x_i + \frac{1}{4}B_1) = \infty$. This is a contradiction.

(4) If possible assume that the image of f is not dense in $\mathbf{C}$. Then there exists a disc $D(a, \epsilon)$ which does not intersct the image so that $|f(z) - a| > \epsilon$ for all z. But then $1/(f - a)$ is a bounded entire function and hence must be constant. This implies that f is also constant which is contradiction.

(5) If possible assume that the sequence of polynomials $\{p_n(z)\}$ converges uniformly on the given annulus to $1/z$. Then for any circle C of radius between 1 and 2, and oriented in the counterclockwise direction,

$$\int_C p_n(z)\, dz \to \int_C \frac{1}{z}\, dz.$$

By Cauchy's theorem, the integral on the left is zero for each n, whereas the integral on the right is $2\pi i$, and thus we have a contradiction.

(6) By means of a conformal map we may think the right half plane as a disc and the imaginary axis as the boundary of the disc. Since on this axis, the linear part $\lambda - z = \lambda - iy$ strictly dominates the exponential part $e^{-z} = e^{-iy}$, By Rouches theorem $\lambda - z$ and $\lambda - z - e^{-z}$ have same number of zeros in the half plane.